Matricial Norms and the Differences between the Zeros of Determinants with Polynomial Elements

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

Submitted by Emeric Deutsch

ABSTRACT

In this paper, we give a method for finding upper bounds for the absolute values of the differences between two latent roots of a lambda-matrix, that is to say, for the differences between two zeros of the determinant of a lambda-matrix. We specialize for complex polynomials.

We intend to determine upper bounds for the absolute values of the differences between any two roots of $\det M(\lambda)=0$, where

$$M(\lambda)=I_s\lambda^n+A_1\lambda^{n-1}+\cdots+A_{n-1}\lambda+A_n,$$

A_j ($j=1,2,\dots,n$) being complex $s\times s$ matrices, i.e., $A_j\in M_{s,s}(\mathbb{C})$. Here and in the sequel I_p denotes the $p\times p$ unit matrix.

It is known [1, 3, 5] that the zeros of $\det M(\lambda)$ are the eigenvalues of the block-companion matrix

$$C=\begin{bmatrix} 0 & I_s & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_s \\ -A_n & -A_{n-1} & -A_{n-2} & \cdots & -A_2 & -A_1 \end{bmatrix}\in M_{ns,ns}(\mathbb{C}).$$

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Clearly, the roots of the equation $\det M_1(\lambda)=0$, where

$$M_1(\lambda)=I_s\lambda^n-A_1\lambda^{n-1}+\cdots+(-1)^{n-1}A_{n-1}\lambda+(-1)^nA_n,$$

are the negatives of the roots of $\det M(\lambda)=0$.

The block-companion matrix of $M_1(\lambda)$ is

$$C_1=\begin{bmatrix} 0 & I_s & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_s \\ (-1)^{n+1}A_n & (-1)^{n}A_{n-1} & (-1)^{n-1}A_{n-2} & \cdots & -A_2 & A_1 \end{bmatrix}\in M_{ns,ns}(\mathbb{C}).$$

According to a result of Rutherford [8; 5, p. 134], the n^2s^2 differences between the eigenvalues of C are the eigenvalues of the matrix

$$C[C_1]=C*I_{ns}+I_{ns}*C_1\in M_{n^2s^2,n^2s^2}(\mathbb{C}),$$

where $X*Y$ denotes the Kronecker product of the matrices X and Y.

We have

$C[C_1]=$

$$\left[\begin{array}{ccccc|c} \Delta C_1 & \Delta I_{ns} & \Delta 0 & \cdots & \Delta 0 & \Delta 0 \\ \Delta 0 & \Delta C_1 & \Delta I_{ns} & \cdots & \Delta 0 & \Delta 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \Delta 0 & \Delta 0 & \Delta 0 & \cdots & \Delta C_1 & \Delta I_{ns} \\ \hline -A_n*I_{ns} & -A_{n-1}*I_{ns} & -A_{n-2}*I_{ns} & \cdots & -A_2*I_{ns} & -A_1*I_{ns}+\Delta C_1 \end{array}\right]\in M_{n^2s^2,n^2s^2}(\mathbb{C}), \tag{1}$$

where

$$\Delta A=\operatorname{diag}\underbrace{(A,A,\ldots,A)}_{s\text{ times}}.$$

If $s=1$ [that is, if $M(\lambda)$ is a polynomial with scalar coefficients], then (1) becomes

$$C[C_1]=\left[\begin{array}{ccccc|c} C_1 & I_n & 0 & \cdots & 0 & 0 \\ 0 & C_1 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_1 & I_n \\ \hline -A_nI_n & -A_{n-1}I_n & -A_{n-2}I_n & \cdots & -A_2I_n & -A_1I_n+C_1 \end{array}\right]\in M_{n^2,n^2}(\mathbb{C}). \tag{2}$$

Let φ denote the matricial norm on $M_{n^2s^2,n^2s^2}(\mathbb{C})$ induced by the column norm[1] $\|\cdot\|_1$ and by the indicated partitioning of $C[C_1]$.

Denoting

$$S = \max|z_i - z_j|,$$

where the maximum is taken over all zeros z_i, z_j of $\det M(\lambda)$, and denoting by $\rho(A)$ the spectral radius of a matrix A, we have [2, 6, 7, 10]

$$S = \rho(C[C_1]) \leqslant \rho(\varphi(C[C_1])). \tag{3}$$

If $s=1$, then we have from (2) and (3)

$$S \leqslant \rho\begin{pmatrix} 1+\alpha & 1 \\ p & \gamma \end{pmatrix}, \tag{4}$$

where

$$\alpha = \max_{i=1,2,\ldots,n-1}\{1+|A_i|, |A_n|\},$$

$$p = \max_{i=2,\ldots,n}|A_i|,$$

$$\gamma = \max_{i=2,\ldots,n-1}\{1+|A_1|+|A_i|, |A_1|+|A_n|\}.$$

If $s>1$, then from (1) we deduce

$$\varphi(C[C_1]) \leqslant \begin{pmatrix} 1+\|C_1\|_1 & 1 \\ P & \|A_1\|_1+\|C_1\|_1 \end{pmatrix},$$

where $P = \max_{i=2,\ldots,n}\|A_i\|_1$, and so (3) becomes

$$S \leqslant \rho\begin{pmatrix} 1+\beta & 1 \\ P & \|A_1\|_1+\beta \end{pmatrix}, \tag{5}$$

where $\beta = \|C_1\|_1 = \|C\|_1 = \max_{i=1,\ldots,n-1}\{1+\|A_i\|_1, \|A_n\|_1\}$.

[1]For $A=(a_{ij}) \in M_{p,p}(\mathbb{C})$, we have $\|A\|_1 = \max_{j=1,2,\ldots,p}\{\sum_{i=1}^{p}|a_{ij}|\}$. Of course, instead of $\|\cdot\|_1$ we can take other matrix norms as well.

REMARK 1. If in the matrix (1) we take the scalar norm $\|\cdot\|_1$ of each of the n^2 blocks, then we obtain

$$S \leqslant 2\beta. \tag{6}$$

REMARK 2. The bound given by (5) may be better than the bounds (4), (7) and (8), where

$$S \leqslant 2\sqrt{(n-1)+\sum_{k=1}^{n}|a_k|^2-\frac{2}{n}|a_1|^2} \qquad [4;\ 5,\ \text{pp. } 13,\ 133] \tag{7}$$

and

$$S \leqslant 2\alpha \qquad [9], \tag{8}$$

if we expand $\det M(\lambda)=0$.

NUMERICAL EXAMPLE. For $\det M(\lambda)=\det(I\lambda^2+A_1\lambda+A_2)$, with

$$A_1=\begin{pmatrix}-1 & 2\\ -6 & -9\end{pmatrix} \quad \text{and} \quad A_2=\begin{pmatrix}0 & -2\\ 12 & 14\end{pmatrix},$$

we have

$$(5):\quad S \leqslant 28.4; \qquad (6):\quad S \leqslant 33.$$

Expanding the determinant, we have

$$\det M(\lambda)=\lambda^4-10\lambda^3+35\lambda^2-50\lambda+24,$$

and we obtain

$$(4):\quad S \leqslant 64; \qquad (8):\quad S \leqslant 102; \qquad (7):\quad S \leqslant 93.32.$$

REMARK 3. Let us now consider the polynomial [5, p. 134]

$$f(z)=z^6-z^5+2z^4+2z^3+2z^2+z-1.$$

We have

$$(4):\quad S \leqslant 5.41; \qquad (8):\quad S \leqslant 6; \qquad (7):\quad S \leqslant 6.25.$$

That is, the inequalities (4) and (8) can give better bounds than (7).

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